Sample Maths Interview Question

Maths & Cryptography Internship

During the interview for the *Maths and Cryptography Internship*, candidates will be asked to work through two mathematical problems with the interviewers. The purpose of this exercise is not to test understanding of a specific piece of maths but is instead designed to assess how you approach and apply maths to new or unfamiliar problems. Candidates are encouraged to think aloud and discuss their answers with the interviewers.

Before starting, the candidate will be shown a short preamble which introduces the problem. This is followed by a series of questions which may be accompanied by additional reading if required. In this sample, there are two sets of additional reading, which should be read before starting questions 2 and 4 respectively (but not before).

Whilst this sample is representative of the style of questions we ask in interviews for the internship, the subject and mathematical content may differ substantially. In addition, there are more questions here than we would expect to be answered in the interview.

We have provided some sample solutions; however, there are many other valid approaches to reach the same answers. Any mathematical, logical solution would be accepted.

Please note that this sample question is provided as reference material for candidates applying for the Maths and Cryptography Internship and may not be representative of our other campaigns.

Magic Squares

A **Magic Square of order $\boldsymbol{n}$** is an $n \times n$ grid containing each of the numbers 1 to n^2 exactly once, such that the numbers in each row, column, and two main diagonals sum to the same value.

For this question, you are reminded that $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

You may use a calculator if you wish.

Questions

Q1: What do the numbers in each row of a Magic Square of order n add up to?

Consider the partially complete 3×3 Magic Square below:

4		
	1	

Q2: Explain why the 5 needs to be in the centre of the grid.

Q3: Use the rules of a Magic Square to complete the grid.

Al-Kishnawi devised a simple algorithm that will populate a Magic Square of **odd order**:

1. Start by placing a 1 in the cell just below the middle square.
2. Continue by placing the next consecutive number one space to the right and one space down from the most recently placed number until the grid is filled.
   a. If you step outside the grid, wrap around onto the opposite side.
   b. If the space where you would insert a number is already filled, place the number two spaces below the most recently placed number instead.

Q4: Demonstrate that this algorithm produces the same Magic Square of order 3 to the one you filled in previously.

Q5: Now use Al-Kishnawi's algorithm to construct a Magic Square of order 5.

Q6: Would this algorithm produce a Magic Square on a grid of even order? Explain your answer.

Q7: Would the result still be a Magic Square if the starting cell was different? Explain your answer.

Sample Solutions

Q1: What do the numbers in each row of a Magic Square of order n add up to?

The sum of all numbers in the grid is $1 + 2 + \cdots + n^2 = \frac{1}{2}n^2(n^2 + 1)$

The rules of a Magic Square dictate that each row sums to the same value, and the sum of all rows is equal to the sum computed above, so we must divide this result by the number of rows to get $\frac{1}{2}n(n^2 + 1)$.

Q2: Explain why the 5 needs to be in the centre of the grid.

The key idea to note for this question is that every cell in the grid belongs to a row/column/diagonal also containing the middle square.

One way in which we could proceed is to do a proof by contradiction. If we were to place a number smaller than 5 in the centre, then we would be unable to place the 1. Suppose we had placed the 4 in the centre rather than 5 – we would need to place a "10" in the square opposite the 1 for the row sum to be correct. Since we are not allowed to place a 10 in the grid, this means that the value in the middle square must be greater than 4. We can use a similar argument to show the middle number must also be less than 6.

Alternatively, we could attempt to construct an equation to find the value of the middle square. Call the value in the middle square x. Consider the row, column, and two diagonals which contain x. Each square in the grid is used once in these, apart from the centre square which is used 4 times. This means that these 4 lines must sum to $45 + 3x$, where 45 is the sum of the numbers 1 to 9. We also know that each row, column, and diagonal sums to the same total, which in the 3x3 case is 15, so these 4 lines must sum to 60, giving us $45 + 3x = 60$. We can rearrange this to get that $x = 5$.

Q3: Use the rules of a Magic Square to complete the grid.

Once we have placed the 5 in the middle square, the remaining values can be inserted by using the fact that each row, column, and two main diagonals must sum to 15. The finished grid is:

4	9	2
3	5	7
8	1	6

Q4: Demonstrate that this algorithm produces the same Magic Square of order 3 to the one you filled in previously.

Follow the algorithm, and check that the resulting square is the same as the one for Q3.

Q5: Now use Al-Kishnawi's algorithm to construct a Magic Square of order 5.

The completed grid is as follows:

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

You may choose to verify your answer by checking a row/column/diagonal to make sure you get the expected sum.

Q6: Would this algorithm produce a Magic Square on a grid of even order? Explain your answer.

There are multiple reasons why the algorithm doesn't work. Attempting to follow the algorithm on a small even grid and reaching a contradiction would be an acceptable answer – for example, the algorithm requires you to start "in the space just below the middle square," which is undefined for a grid of even order.

Alternatively, if we did have a starting point, we can show that the algorithm would only fill in half the grid. Notice that as we fill a column, the next cell we fill is 2 cells below the previous one, so to fill the entire grid the number of rows needs to be coprime to 2.

Q7: Would the result still be a Magic Square if the starting cell were different? Explain your answer.

Note that this question is not asking us to prove that the algorithm works. Instead, we may assume that the algorithm works as described and we need to show that any other choice of starting position would cause the algorithm to fail.

Let us start by identifying which properties still hold true. The algorithm will still fill every cell in the grid and the sum of each row and column will be preserved. We can justify this by noticing that the way in which the grid is filled-in behaves topologically as a torus. What this means is that as we traverse the grid, the left and right edge behave as if they are connected; similarly, the top and bottom edge. In practice, this means that if we were to shift the starting position one cell to the left, the resulting grid would look identical except the leftmost column will have moved to become the rightmost column. The same is true for rows and starting one cell higher/lower. The process of moving rows and columns does not change their sums, so the row and column sums are preserved under changes to the starting cell.

This means that the only thing left to consider is the sum of the two main diagonals. The leading diagonal (starting at the top left corner) will always consist of a run of five consecutive numbers. The only exception to this is in the case where the starting cell is itself on the main diagonal, in which case the numbers are still consecutive mod n^2. The only run of consecutive numbers with the correct sum is the one centred on the median, so the median must be placed on this diagonal.

We also need to consider the counter diagonal. All the diagonals in this direction are arithmetic sequences that increase by n mod n^2. Because there are an odd number of values, the sum of the arithmetic expression can be expressed as $n \cdot x$, where x is the middle number of the sequence. Comparing this to the row sum we calculated earlier, we get $n \cdot x = \frac{1}{2}n(n^2 + 1)$, telling us that x must be the median.

Because the median must be placed on both diagonals, it must be in the centre square. Since we are assuming that the algorithm (as described) is correct, it must always place the median in the middle square as required. Hence, any other starting position will place the median in a different square, so cannot give a valid Magic Square.